8. Differentiation

- **Derivatives**
  - Suppose *f* is a real-valued function and *a* is a point in its domain of definition. The derivative of *f* at *a* [denoted by $f'(a)$] is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$, provided the limit exists.

Derivative of $f(x)$ at a is denoted by $f'(a)$.

  - Suppose *f* is a real-valued function. The derivative of $f \left\{ \text{denoted by } f'(x) \text{ or } \frac{d}{dx}[f(x)] \right\}$ is defined as

$$\frac{d}{dx}[f(x)] = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, provided the limit exists.

This definition of derivative is called the first principle of derivative.

Example: Find the derivative of $f(x) = x^2 + 2x$ using first principle of derivative.

Solution: We know that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore f'(x) = \lim_{h \to 0} \frac{(x+h) + 2(x+h) - (x^2 + 2x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2hx + 2x + 2h - x^2 - 2x}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2hx + 2h}{h}$$

$$= \lim_{h \to 0} (h + 2x + 2)$$

$$= 0 + 2x + 2 = 2x + 2$$

$$f'(x) = 2x + 2$$

- **Derivatives of Polynomial Functions**

For the functions u and v (provided u' and v' are defined in a common domain),

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  - $(u \pm v)' = u' \pm v'$
  - $(uv)' = u'v + uv'$ (Product rule)
  - $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ (Quotient rule)
- **Derivatives of Trigonometric Functions**
  - $\frac{d}{dx}(x^n) = nx^{n-1}$ for any positive integer n
  - $\frac{d}{dx}\left(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\right) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-1} + \ldots + a_1$

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$

Example: Find the derivative of the function $f(x) = \left(3x^2 + 4x + 1\right) \cdot \tan x$

Solution: We have,

fx=3x2+4x+1.tan xDifferentiating both sides with respect to x, f'x=3x2+4x+1.ddxtan x+tan x .ddx3x2+4x+1 f'x=3x2+4x+1.sec2x+tan x6x+4f'x=3x2+4x+1.sec2x+6x+4tan x

The derivatives of exponential functions are as follows:

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$

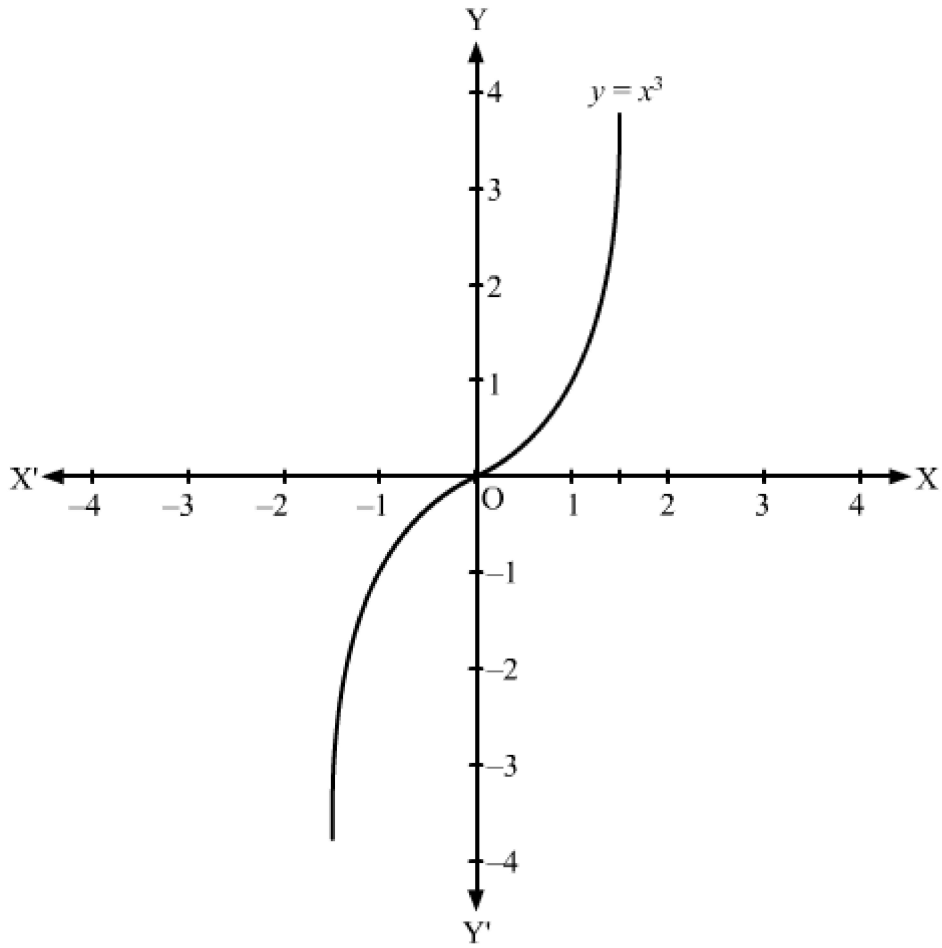
- **Mean value theorem:**
  If $f: [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

  Example: Verify Mean Value Theorem for the function:
  $f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

  Solution:
  $f(x) = 2x^2 - 17x + 30$
  $\therefore f'(x) = 4x - 17$
  The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.
  Also, $f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$
  and, $f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$
  $\therefore f\left(\frac{5}{2}\right) = f(6)$

  Now, $\frac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$

  According to Mean Value Theorem (MVT), there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$.
  $\therefore 4c - 17 = 0$
  $\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$



Therefore, M.V.T is verified.